# Anchored Spiral Waves in the $\theta$ Model


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# Background

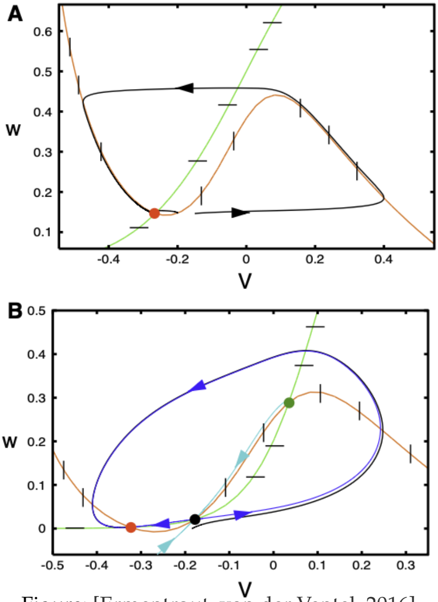



Figure: [Ermentrout, van der Ventel, 2016]

**A: Type II excitability**

1. FitzHugh-Nagumo model.
2. Hodgkin-Huxley model.
3. One fixed point.
4. Loses stability via a Hopf bifurcation.
5. Stable limit cycle (traveling pulses).

**B: Type I excitability**

1. Ermentrout, Rinzel, 1998
2. Izhikevich, 2007
3. Three fixed points.
4. Saddle-node on invariant circle (SNIC).
5. Unstable manifolds of the saddle are strongly attracting (even without fast-slow time scales).

# The $\theta$ model

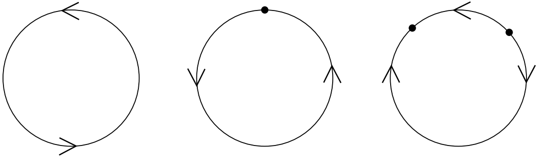
Reaction-diffusion equation on an annulus with Neumann B.C.

$$u_t = \Delta u + f(u;\mu), \quad x \in \Omega,$$

$$\partial_\nu u = 0 \quad \text{on } \partial\Omega,$$

$$\Omega = \{x \in \mathbb{R}^2 : R_- < |x| < R_+\}, \quad R_+ \leq \infty$$

$$f(u;\mu) : 2\pi\text{-periodic, e.g. } f(u) = 1 + \mu\cos(u).$$



$\mu < 1$ $\mu = 1$ $\mu > 1$

- $u_t = 1 + \mu\cos(u)$: normal form of Type I excitability.
- Saddle-node on invariant circle (SNIC) bifurcation at $\mu = 1$.

[Ermentrout, Kopell, 1986]

# Polar coordinates and winding numbers

In fact, $u : \Omega \to S^1_{2\pi}$ with a degree $\ell \in \mathbb{Z}$:

$$\begin{aligned} u_t &= u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi} + f(u;\mu), \\ \partial_r u|_{r=R_-,R_+} &= 0, \quad r \in \{R_-, R_+\},\ \varphi \in [-\pi, \pi], \\ \partial_\varphi u(r,0) &= \partial_\varphi u(r, 2\pi), \\ u(r,0) &= u(r,2\pi) + 2\pi\ell. \end{aligned}$$

- Continuity of $u$: $u|_\Gamma \to S^1$ has degree $\ell$ for any curve $\Gamma$ with winding number 1 around the core $\{|x| < R_-\}$.
- Degree: *The Geometry of Biological Time* [Winfree, 1980]
- Non-local model (spiral wave chimera) with a core: [Shima, Kuramoto, 2004], [Martens, Laing, Strogatz, 2010]

# Relative equilibria

Look for rotating waves of the form $u(r, \varphi, t) = u(r, \varphi - \omega t)$.
Substitute $u = v - \varphi$:

$$\begin{aligned} \Delta_{r,\varphi} v - \omega v_\varphi - \omega \ell + f(v - \ell\varphi; \mu) &= 0, \\ v(r, \varphi + 2\pi) - v(r, \varphi) &= 0, \\ v_r\big|_{r=R_-, R_+} &= 0. \end{aligned}$$

**Theorem (Existence on bounded domains)**

*For $R_+ < \infty$ and $\ell \neq 0$ and arbitrary $\mu$ there exists a solution $(u, \omega)$ to the equations. Moreover, $u$ is strictly increasing in $\varphi$. The solution is asymptotically stable.*

# Existence – proof sketch

$$\Delta_{r,\varphi} v - \omega v_\varphi - \omega \ell + f(v - \ell\varphi; \mu) = 0,$$
$$v(r, \varphi + 2\pi) - v(r, \varphi) = 0,$$
$$v_r\big|_{r=R_-, R_+} = 0.$$

**Proof.**

Take $\ell = 1$ for simplicity. Global homotopy for $\tau \in [0, 1]$:

$$f(u; \tau) = \tau f(u) + (1 - \tau) \fint f$$

where $I$ is the set of $\tau$ on which the equation has a monotone solution.

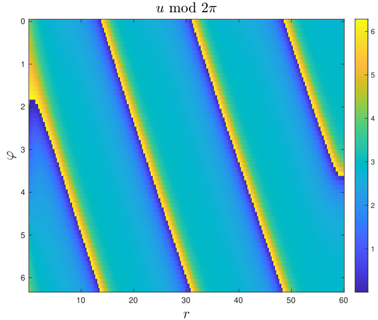
Show (i) $0 \in I$; (ii) $I$ is closed; (iii) $I$ is open.

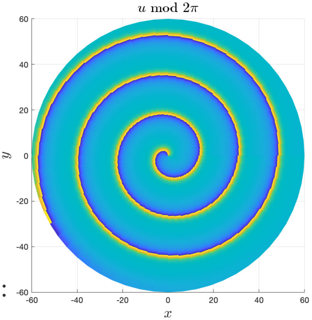
(ii) A-priori estimates, comparison/maximum principles.

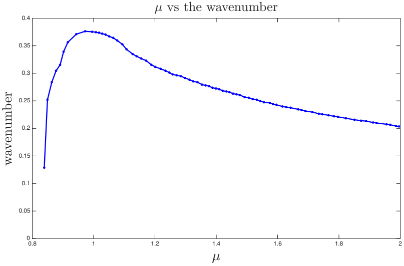
(iii) Implicit function theorem (IFT).

Existence for oscillatory media ($\mu < 1$) and excitable media ($\mu > 1$). □

# Rotating waves are spirals




Need:

1. Existence theorem for $R_+ = \infty$.
2. Numerics for $R_+ = \infty$.
3. Archimedian spiral?

   $\varphi(r) = C_1 r + C_2 \log(r) + \mathcal{O}(1/r)$
4. Wavenumber selection.

   [Ermentrout, van der Ventel, 2016] on thin annulus

# Numerics for $R_+ = \infty$ – farfield-core

For simplicity, we will take $\ell = 1$ in the rest of the talk.

Decompose $v$ into a core part and a wave train part:

$$v = v_{\text{core}} + \chi \cdot v_{\text{wt}}$$

1. $\chi$: smooth cut-off function. For $R_{\text{mid}} \in (R_-, R_+)$,
$$\begin{cases} \chi(r) = 0, & r < R_{\text{mid}} - 1 \\ \chi(r) = 1, & r > R_{\text{mid}} + 1 \end{cases}$$
2. $v_{\text{core}}$: PDE solution localized near the core,
$v_{\text{core}} \to 0$ as $r \to R_+$.
3. $\chi \cdot v_{\text{wt}}$: wave train solution that captures the far-field behavior,
$\chi \cdot v_{\text{wt}} = 0$ as $r = R_-$.

[Morrissey, Scheel, 2015], [Lloyd, Scheel, 2016], [Dodson, Sandstede, 2019], [Dodson, Lewis, 2022]

# Wave train – theory

Substitute $u(r, \varphi, t) = u(kr - \omega t)$ and take $r \to \infty$ limit.

Wave train solutions satisfy the ODE with variable $\psi := kr - \omega t$:

$$\begin{aligned} k^2 u_{\psi\psi} - \omega u_{\psi} + 1 + \mu \cos(u) &= 0, \\ u(\psi + 2\pi) - u(\psi) &= 2\pi \end{aligned}$$

or the periodic version in $v := u + \psi$:

$$\begin{aligned} k^2 v_{\psi\psi} - \omega v_{\psi} - \omega + 1 + \mu \cos(v - \psi) &= 0, \\ v(\psi + 2\pi) - v(\psi) &= 0. \end{aligned}$$

- Obtain $v$ as a 1D BVP.
- Interpolate it onto the 2D domain via $v_{\mathrm{wt}}(r, \varphi) = v(kr + \varphi)$.

Ignore potential log shift and $\mathcal{O}(1/r)$ residuals.

# Wave train - asymptotics for small $\mu$

Write as a first-order system with equilibrium $(v, \mu, \hat{\omega}) = (\text{const}, 0, 0)$:

$$v_\psi = \frac{1}{k} w$$
$$w_\psi = \frac{1}{k}\left[\frac{1}{k}(\hat{\omega}+1)v_\psi + \hat{\omega} - \mu\cos(v-\psi)\right]$$

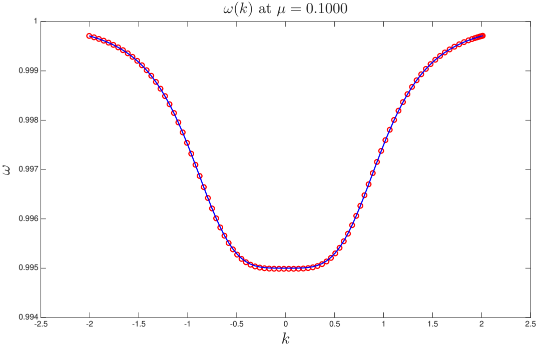
For $0 < \mu \ll 1$, we can expand $v$, $w$, and $\hat{\omega} := \omega - 1$ in $\mu$ as

$$v(0) = 0$$
$$w(0) = w_0 + w_1\mu + w_2\mu^2 + \cdots$$
$$v(r,\psi) = v^0(\psi) + \mu v^1(\psi) + \mu^2 v^2(\psi) + \cdots$$
$$w(r,\psi) = w^0(\psi) + \mu w^1(\psi) + \mu^2 w^2(\psi) + \cdots$$
$$\hat{\omega} = \hat{\omega}_0 + \mu\hat{\omega}_1 + \mu^2\hat{\omega}_2 + \cdots.$$

# Nonlinear dispersion relation for small $\mu$

$$\omega = 1 - \frac{\mu^2}{2(1+k^4)} + \mathcal{O}(\mu^4).$$

In fact, $\omega \sim \omega_0 + \omega_2(\mu)\, k^4 + \mathcal{O}(k^6)$ for $k \sim 0$ (weak nonlinear dispersion).




Blue: theoretical prediction; Red: numerical computation.

[Ermentrout, Rinzel, 1981] on 1D ring

# Spirals via farfield-core

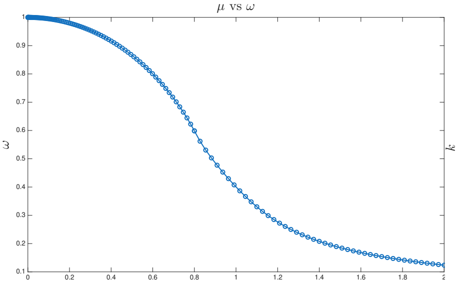
Combine 2D PDE and the wave train ODE, coupled through $\omega$:

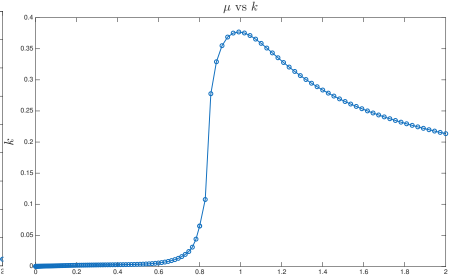
$$
\begin{aligned}
\Delta_{r,\varphi} v - \omega v_\varphi - \omega \ell + f(v - \ell\varphi; \mu) &= 0, \\
v(r, \varphi + 2\pi\ell) - v(r, \varphi) &= 0, \\
v_r\big|_{r=R_-,R_+} &= 0, \\
k^2 v^{\mathrm{wt}}_{\psi\psi} - \omega v^{\mathrm{wt}}_\psi + 1 + \mu\cos(v^{\mathrm{wt}} - \psi) &= 0, \\
\chi\langle v, v^{\mathrm{wt}}_\psi \rangle &= 0.
\end{aligned}
$$

Substitute the farfield-core ansatz $v = v_{\mathrm{core}} + \chi \cdot v_{\mathrm{wt}}$.

Solve for $v_{\mathrm{core}}$, $v_{\mathrm{wt}}$, $\omega$ , $k$ simultaneously.

# Farfield-core – numerical results




Frequency $\omega$ and wavenumber $k$ selected by the spiral wave.

Empirically for $\mu \sim 0$,

$$\omega(\mu) \sim 1 - \omega_2 \frac{\mu^2}{2}, \quad k(\mu) \sim k_2 \mu^2$$

Work in progress:

1. Determine powers theoretically, and confirm with numerics.
2. Existence of spirals for $\mu \ll 1$.

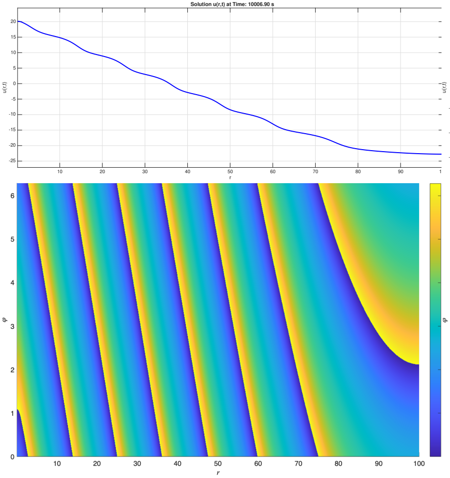
# Toy problem: Spirals via 1D sources

$$u_t = u_{rr} + 1 + \mu \cos(u) + \gamma \chi_{[0,1]}, \quad r \in (R_-, \infty), \quad u_r|_{r=R_-} = 0.$$

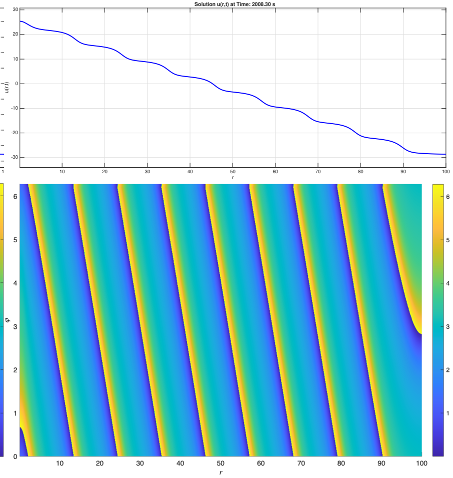
1. No curvature effect.
2. Inhomogeneity $\gamma \chi_{[0,1]}$ with support in $[0, 1]$ and strength $\gamma$.
3. To obtain 2D rotating waves, identify $\varphi = -\omega t$ where $\omega = 2\pi/T$, $u(t + T) = u(t) + 2\pi$.
4. One can homotopy from 1D sources to 2D spirals via

$$\tau \cdot \left( \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi} \right) + (1 - \tau) \cdot \gamma \chi_{[0,1]}$$

# Spirals from 1D sources – numerics

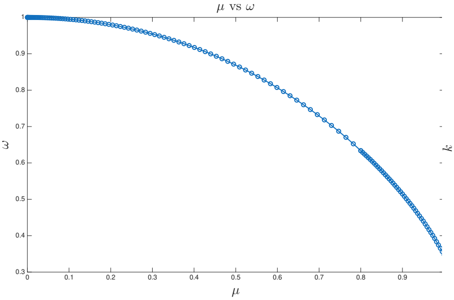


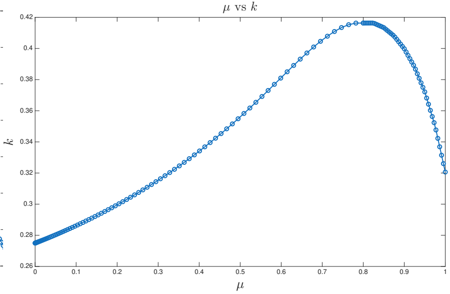

(a) $\mu = 0.5$, $\omega = 0.8675$.




(b) $\mu = 0.8$, $\omega = 0.6888$.

# 1D sources with farfield-core



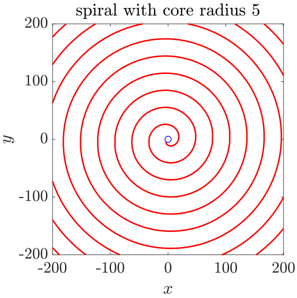



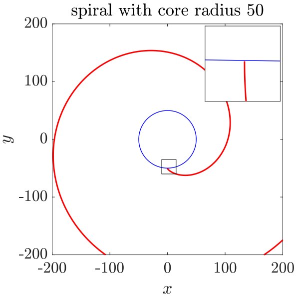

1D sources select $\omega$ and $k$ in a different way from curvature effects.

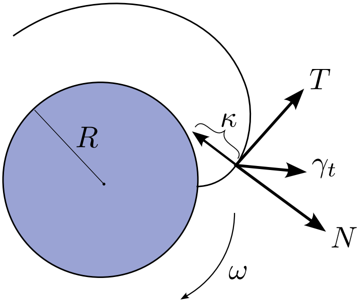
For $\gamma = 0.5$ and $\mu \sim 0$, we have empirical asymptotics

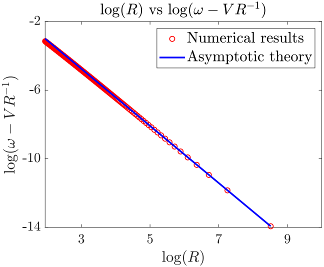
$$\omega(\mu) \sim 1 - \frac{\mu^2}{2}, \quad k(\mu) \sim 0.275 + k_1\mu.$$

# Spiral via geometric flows – numerics



# Spiral waves via geometric flows

Curvature driven flow $c = V + D\kappa$. $V, D > 0$; $c$: normal velocity.

Spiral as a planar curve $\gamma = \{(r, \Phi(r,t))\}$ with

$$\Phi_t = \frac{Dr\Phi_{rr} - V(1 + r^2\Phi_r^2)^{3/2} + Dr^2\Phi_r^3 + 2D\Phi_r}{r(1 + r^2\Phi_r^2)},$$

$$\Phi_r = 0 \quad \text{at } r = R_- \qquad R_+ = \infty$$

**Theorem (Li, Scheel, 2024)**

1. *(Existence) For all $D, V, R_- > 0$, there exists an outward rotating asymptotically Archimedean spiral $\gamma_0$ with frequency $\omega = \omega_{sp}(D, V) > 0$.*
2. *(Asymptotics) For fixed $D, V > 0$, and $R \gg 1$, we have the expansion*
$$\omega_{sp} = VR^{-1} - \sigma_0 2^{1/3} D^{2/3} V^{1/3} R^{-5/3} + O(R^{-7/3}), \quad \sigma_0 = 1.01879297\ldots$$
3. *(Spirals are Lyapunov stable) All curves that are sufficiently close to the solution will stay close to it for all times.*

**Proof.**

- GSPT & shooting method for existence and asymptotics.
- comparison principle for stability.

# Spiral instabilities

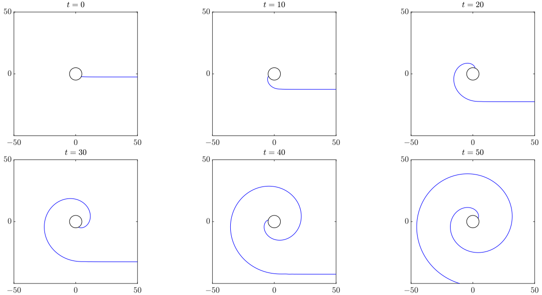
Not all spirals are stable. For example, with $R_-, R_+$ sufficiently large,

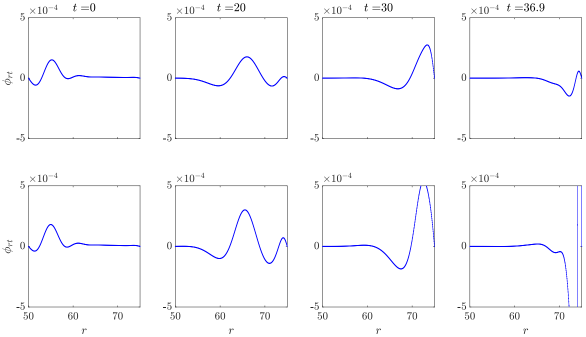
$$c = V + D_2\kappa - D_4\kappa_{ss}$$

**Results (Cortez, Li, Mihm, Xu, Yu, Scheel, 2025)**

1. *Fix $V > 0, D_4 > 0$, and $D_2$, then for appropriate B.C. at $r = R_-$ and all $R_- \gg 1$, there exists a rigidly rotating spiral wave solution.*
2. *Fix $V, D_4 > 0$, and $R_- \gg 1$, then for $D_2 > 0$, spirals are stable (linearization has no unstable eigenvalues).*
   *When $D_2$ decreases past $D_2^{\text{crit}}(R_-, D_4, V) < 0$, the spiral wave undergoes a Hopf bifurcation with an eigenfunction that grows super-exponentially as $r \to \infty$.*

# Spiral instabilities – numerics



# Summary & Questions

**Summary:**

- Anchored spiral waves in the $\theta$ model (oscillatory & excitable).
- Existence of rotating waves on bounded domains.
- Numerical results from farfield-core matching.
- Dispersion relation for wave trains.
- Wavenumber & frequency selection by curvature effects and 1D sources.
- Spirals from 1D sources and curvature-driven geometric flows.

**https://nanli.co**

**Open questions:**

- Existence for $R_+ = \infty$ ($\mu \sim 0$).
- Stability for $R_+ = \infty$.
- Expansion of frequency $\omega(\mu)$ and wavenumber $k(\mu)$ for $\mu \sim 0$.