Anchored Spirals in Sharp-Interface and Phase Oscillator Models

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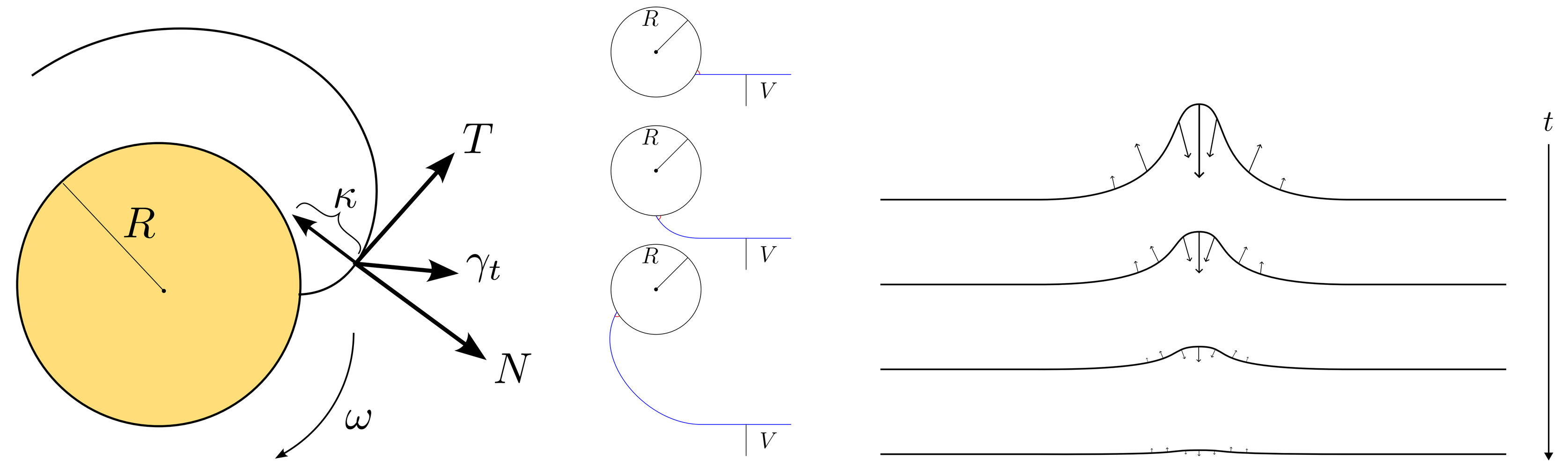
§I. Sharp-Interface Model [Li, Scheel, 2024]



- Curvature flow: $c = V - D\kappa$
- c: Normal velocity
- V: Propagation velocity of the straight-line interface (curling-up of straight line: $c = V$)
- D: Line tension (curve-shortening flow: $c = -D\kappa$)
- The wave front is a planar curve written in the **polar coordinate**

$$\gamma(t,r) = (r\cos(\Phi(t,r)), r\sin(\Phi(t,r)))$$

Evolution equation:

$$\Phi_t = \frac{Dr\Phi_{rr} - V(1+r^2\Phi_r^2)^{3/2} + Dr^2\Phi_r^3 + 2D\Phi_r}{r(1+r^2\Phi_r^2)}$$

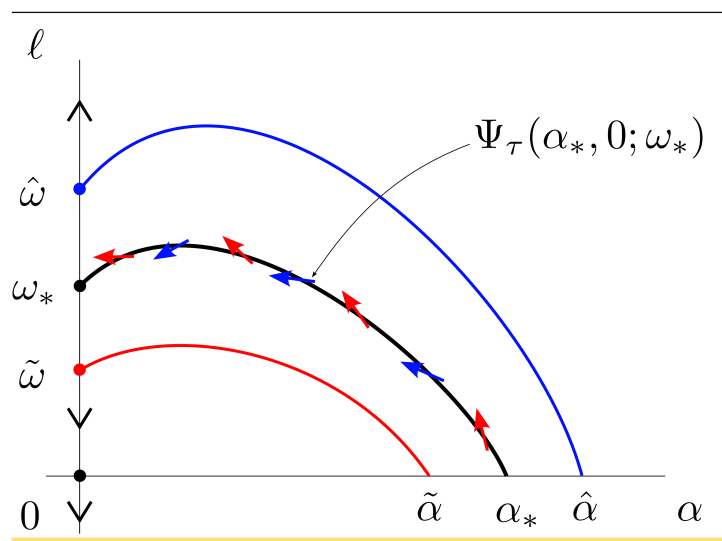
ODE from rotating wave ansatz $\Phi(t,r) = \phi(r) - \omega t$:

$$\begin{cases} \ell = \phi_r \\ \alpha = 1/r \\ \tau = (r^3 - R^3)/3 \end{cases} \Rightarrow \begin{cases} \ell_\tau = -\frac{\omega}{D}(\alpha^2+\ell^2) + \frac{V}{D}(\alpha^2+\ell^2)^{3/2} - 2\alpha^3\ell - \alpha\ell^3 \\ \alpha_\tau = -\alpha^4 \end{cases}$$

§I. Theorem 1: Existence of rigidly rotating spirals

Fix $D, V > 0$ and let $(\alpha(\tau;\omega), \ell(\tau;\omega))$ denote the solution of the ODE with initial condition $(\alpha(0), \ell(0)) = (\alpha_*, 0)$ and parameter ω. Then there exists, for every $\alpha_* > 0$, a unique ω_* such that $\lim_{\tau\to\infty} \ell(\tau;\omega_*) = \omega_*/V$. Moreover, ω_* is strictly increasing in α_*.

§I. Proof of Theorem 1: Shooting argument



- Unique correspondence between core radius R and the angular velocity ω
- Solutions are Archimedean spirals in the farfield ($r \to \infty$):

$$\phi(r) = kr + \text{const}\cdot\log r + \mathcal{O}(r^{-3})$$

- Wavenumber $k = \omega/V$

§I. Theorem 2: Asymptotic expansion in the large-core limit

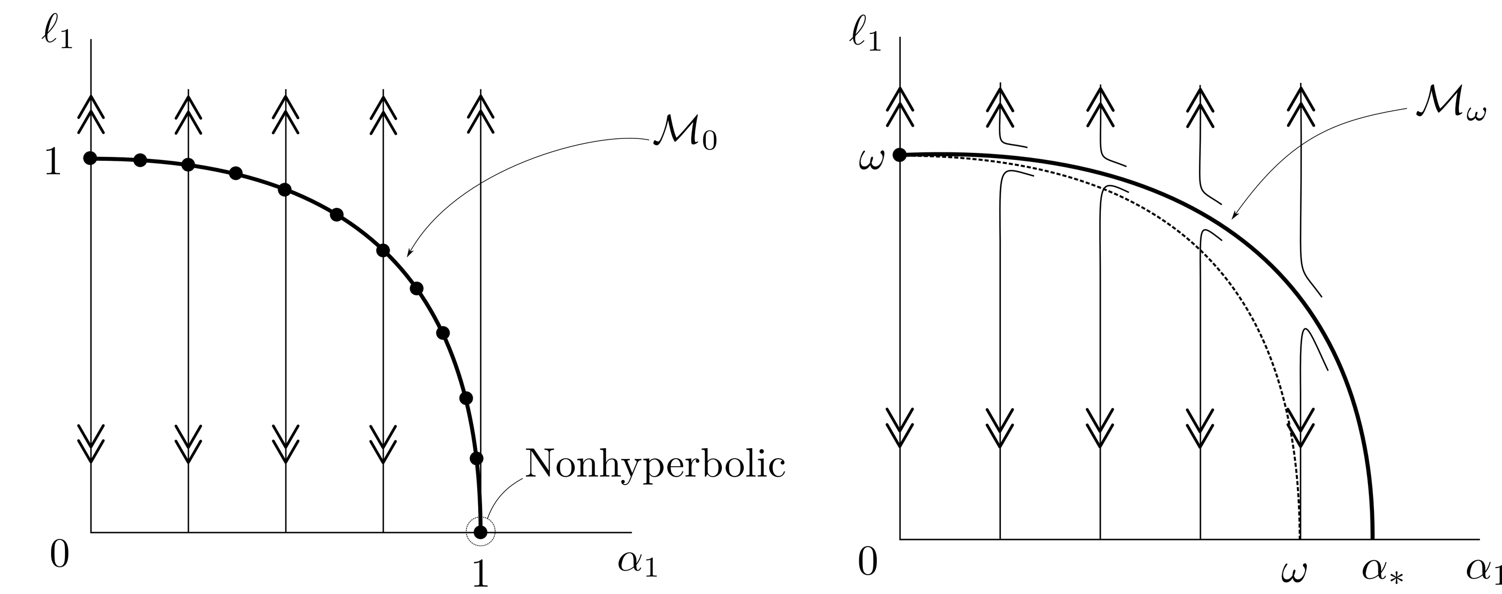
Given $\alpha_* > 0$, let ω_* and $\ell = \lambda(\alpha)$ be the solution from Theorem 1. We then have the expansions

$$\omega_* = V\alpha_* - \sigma_0\sqrt[3]{2D^2V}\alpha_*^{5/3} + \mathrm{O}(\alpha_*^{7/3}),$$

$$\lambda(\alpha) = \sqrt{\frac{\omega_*^2}{V^2} - \alpha^2} + \mathrm{O}(\omega\alpha), \quad \text{for } \alpha < (1-\delta)\frac{\omega_*}{V} \text{ and some } \delta > 0,$$

where $\sigma_0 = 1.01879297\ldots$ is determined by the first zero of the derivative of the Airy function, that is, $\mathrm{Ai}'(-\sigma_0) = 0$, $\mathrm{Ai}'(-\sigma) > 0$ for $\sigma < \sigma_0$.

§I. Proof of Theorem 2: Fenichel's Theorem, Krupa & Szmolyan



§I. Theorem 3: Stability of Solutions

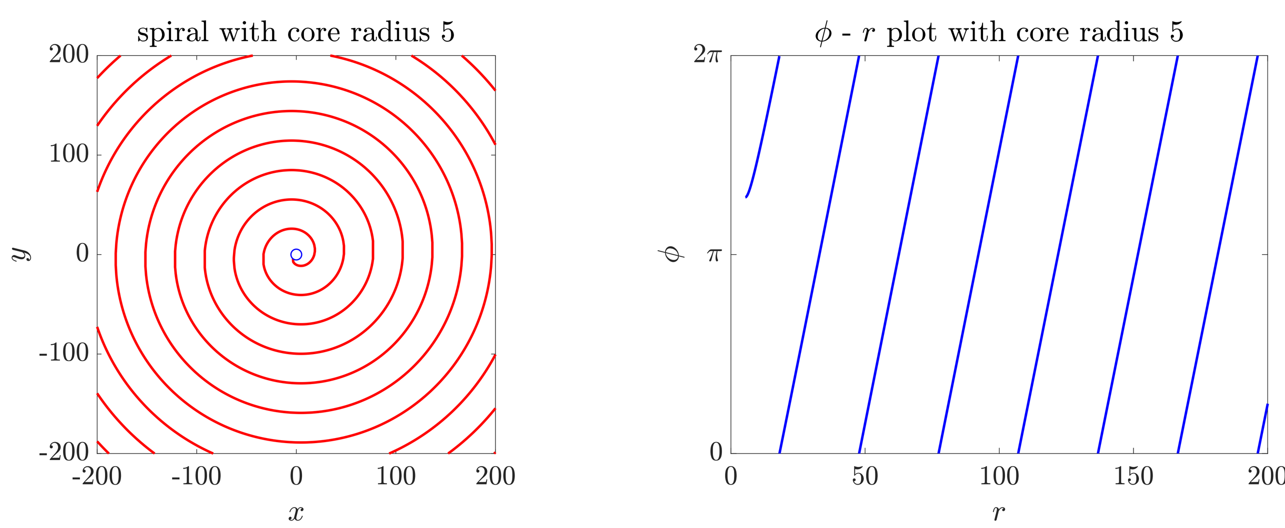
For all $\varepsilon > 0$, there exists $\delta > 0$ so that for all $\varphi \in C^2_{\mathrm{loc}}([R,\infty))$ with

$$\sup_r(|r^{-2}\varphi| + |r^{-1}\varphi_r| + |\varphi_{rr}|) < \delta, \quad \varphi_r(R) = 0,$$

we have that the solution $\Phi(t,r)$ with initial condition $\phi_*(r) + \varphi(r)$ to the evolution equation satisfies $\|\Phi(t,\cdot)\|_{C^0} < \varepsilon$ for all $t > 0$.

1. Local well-posedness and regularity: there exists a unique global solution to the evolution equation with the initial data $\varphi_r(R) = 0$ and takes the form $\varphi_t \sim \frac{1}{r^2}\varphi_{rr} + \varphi_r$ as $r \to \infty$.
2. A priori bounds on Φ and Φ_r from super- and sub-solution (the comparison principle).

§I. Numerical Computation: Archimedean Spirals



§II. Transverse Instability [Cortez, Li, Mihm, Xu, Yu, Scheel, 2025]

Curvature Flow: $c = V + D_2\kappa - D_4\kappa_{ss}$

- s: Arclength
- Geometric singular perturbation at $D_2 = D_4 = 0$
- Rigidly rotating spiral for large core radius $R_\mathrm{i} \gg 1$
- Hopf bifurcation as D_2 changes sign, exhibiting instabilities

Evolution Equation:

$$\begin{aligned}\Phi_t = &-\Phi_{rrrr}\frac{D_4}{M^4} + \Phi_{rrr}\frac{D_4}{M^8}\left(6r^3\Phi_r^4 + \Phi_{rr}\left(10r^4\Phi_r^3 + 10r^2\Phi_r\right) + 2r\Phi_r^2 - \frac{4}{r}\right)\\ &+ \Phi_{rr}^3\frac{D_4}{M^8}\left(-15r^4\Phi_r^2 + 3r^2\right) + \Phi_{rr}^2\frac{D_4}{M^8}\left(-21r^3\Phi_r^3 + 33r\Phi_r\right)\\ &+ \Phi_{rr}\frac{D_4}{M^8}\left(-r^4\Phi_r^6 - 19r^2\Phi_r^4 + 36\Phi_r^2\right) + \Phi_{rr}\frac{D_2}{M^2}\\ &- \Phi_r^7\frac{3D_4r^3}{M^8} - \Phi_r^5\frac{17D_4r}{M^8} + \Phi_r^3\frac{\left(D_2M^6r^2 + 4D_4\right)}{M^8r} + \Phi_r\frac{2D_2}{M^2r} - \frac{MV}{r}.\end{aligned}$$

§II. Transverse Instability: Theorems

Fix $V > 0$, $D_4 > 0$, and D_2. Given "compatible boundary conditions" at $r = R_\mathrm{i}$ and all $R_\mathrm{i} \gg 1$.

Result 1: There exists a rigidly rotating spiral wave solution with frequency

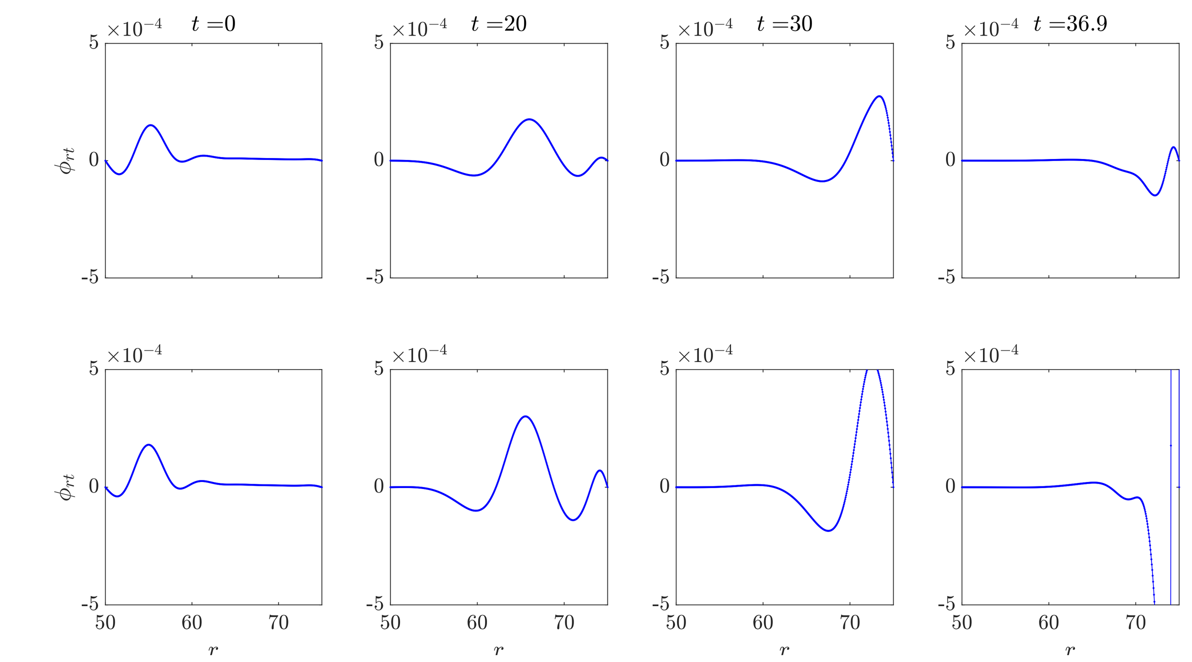
$$\omega = \frac{V}{\sin(\vartheta_\mathrm{i})}R_\mathrm{i}^{-1} + \mathrm{O}(R_\mathrm{i}^{-2}),$$

where $\vartheta_\mathrm{i} \in (0, \pi/2)$ is the contact angle between the curve and the inner circle.

Result 2: There exists a $D_2^{\mathrm{crit}}(R_\mathrm{i}, D_4, V) = -\sqrt[3]{\frac{81}{4}(7\sqrt{7} - 17)D_4V^2\cot^2(\vartheta_\mathrm{i})} < 0$ such that

- No unstable eigenvalues for $D_2 > D_2^{\mathrm{crit}}$.
- Hopf instability with super-exponential growth as $r \to \infty$ for $D_2 < D_2^{\mathrm{crit}}$.

An initial Gaussian perturbation is advected to the outer boundary. Both time series are for $R_\mathrm{i} = 50$, $R_\mathrm{o} = 75$, $\vartheta_\mathrm{i} = \pi/2 - 0.1$, $D_4 = V = 1$, so that $D_{2,\mathrm{crit}} \sim -0.67$. Top: $D_2 = -0.5$; Bottom: $D_2 = -0.6$.



§III. Phase Oscillator Model [Work in Progress]

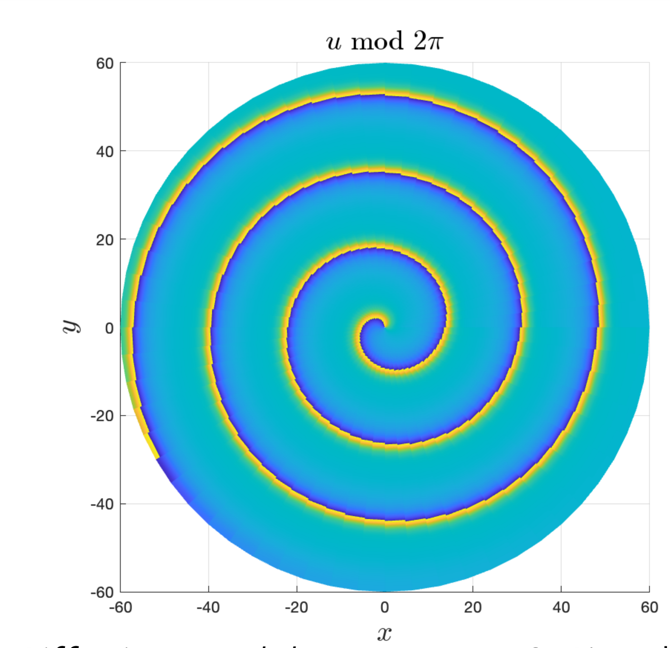
Reaction-Diffusion Equation on $\Omega = \{R_- \le |x| \le R_+\} \subset \mathbb{R}^2$

$$\begin{cases} u_t = \Delta_{r,\varphi}u + f(u;\mu), & x \in \Omega, \\ \partial_\nu u = 0 & x \in \partial\Omega. \end{cases}$$

Relative equilibrium via corotating frame $\phi = \varphi - \omega t$

$f(u;\mu)$: 2π-periodic in u

$$\begin{cases} 0 = \Delta_{r,\phi}u - \omega u_\phi + f(u;\mu), \\ 0 = u(r,\phi+2\pi) - u(r,\phi) - 2\pi\ell, \\ 0 = u_r|_{r=R_-,R_+} \end{cases}$$



Waves in a Simple, Excitable or Oscillatory, Reaction-Diffusion Model Ermentrout & Rinzel 1981.

§III. Existence of Spirals on Bounded Annulus

For $\ell \neq 0$, there exists a solution (u, ω) to the BVP. Moreover, u is strictly increasing in ϕ.

Proof: Global homotopy: $f(u;\tau) = \tau f(u) + (1-\tau)\int f$.

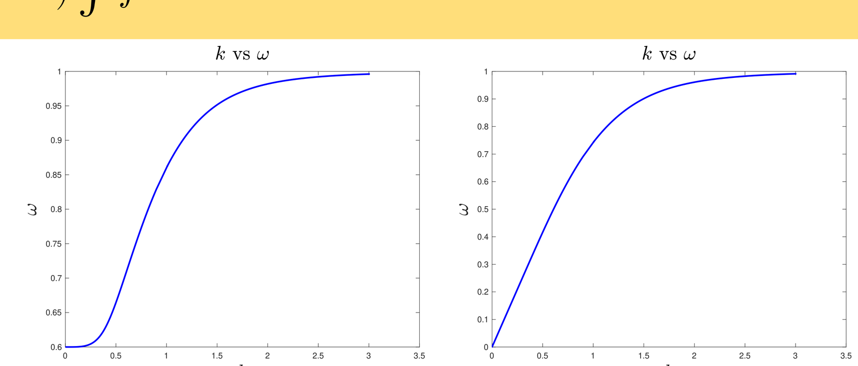
More Open Problems:

1. Existence of spirals on unbounded annulus

$$u_{\text{bounded annulus}} \xrightarrow{\text{loc, unif}} u_{\text{unbounded annulus}}$$

2. Wave train selection

$$\omega = \omega(k;\mu), \qquad k: \text{wavenumber}$$



Conjectured in [Ermentrout & Rinzel 1981]